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LETTER TO THE EDITOR

Discrete compactons: some exact results

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Abstract

In this letter, we use the method of constructing exact solutions on lattices proposed by Kinnersley and described in Schmidt (1979 *Phys. Rev.* B **20** 4397), to obtain exact compacton solutions in discrete models. We examine the linear stability of such solutions, both for the bright compacton and for the dark compacton cases. We focus on a 'quantization condition' that the width of the profile should satisfy. We also use this quantization condition to examine the possibility of compact coherent structures travelling in discrete settings. Our results are obtained for sinusoidal profiles and then generalized to elliptic functions of arbitrary modulus. The possibility of multi-compacton solutions is considered.

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1. Introduction

One of the most interesting recent developments in the theory of dispersive nonlinear partial differential equations (PDEs) has been the discovery of compactons [1]. These are compact (usually described by powers of trigonometric functions) solutions of partial differential equations that have the feature of nonlinear dispersion [1, 2]. Even though these nonlinear waves and their comparison with the customary exponentially localized solitary wave structures of dispersive nonlinear PDEs [3] are of substantial mathematical interest, one of the main concerns about such waves pertains to their relevance in physically realistic models.

Another interesting issue for such structures is whether they are supported in discrete settings, which are very often relevant in realistic problems. In these lattice contexts, there has been a large amount of work on the behaviour of discrete solitons (often referred to as intrinsic localized modes) in the past decade; see, e.g., the reviews [4] for a summary of recent theoretical and experimental results and applications. On the other hand, there are some sparse results on the existence and stability of exact discrete compactons in some mean field ferromagnetic and other more formal models [5, 6], and on the existence of compact discrete breathers (displaying a decay more rapid than the exponential one) in some classes of models [7].

Here we explore an alternative direction, partially in the spirit of the works of [8], motivated by the original idea of Kinnersley, implemented in [9]. In particular, in [9] an inverse method was proposed for constructing (discrete) models supporting exact static and/or travelling nonlinear waves (in that reference only of the exponentially localized variety). This technique, however, can, in principle, be applied also for the construction of compacton-bearing models, through an appropriate selection of initial (solution) ansatz. However, further careful examination of the idea suggests that care should be exercised to enforce the appropriate cut-off conditions, so as to preserve the compact support of the relevant structure. This imposes an important quantization condition that has a number of consequences. It quantizes the values of the width parameter of the relevant discrete compacton. Furthermore, it yields insights into the issue of propagation of such compact structures in discrete lattices. These are the issues that are examined in the present work both in cases of bright and dark discrete compactons. Given such exact solutions, their linear stability is examined and their properties are clarified. These results are also generalized to elliptic functions (the natural generalizations of the initial trigonometric ansatz) and the behaviour of the relevant models is examined in the latter case as a function of the elliptic modulus.

Section 2 presents the main bright soliton solutions, while section 3 briefly discusses dark solutions. Section 4 provides insights into travelling, while section 5 presents the generalization to elliptic functions. Finally in section 6, we summarize our findings and conclude.

2. Trigonometric solutions: bright case

To be specific, we will focus here, in particular, on Klein–Gordon-type models with (as is typical) nearest neighbour interaction in the form

$$\ddot{u}_n = g(u_n)(u_{n+1} + u_{n-1}) + f(u_n). \tag{1}$$

The overdot in equation (1) denotes temporal derivative. The non-constant (in contrast to the standard models of harmonic coupling and linear dispersion [4]) function $g(u_n)$ ensures the presence of nonlinear dispersion which is critical for the existence of compactly supported solutions. $f(u_n)$ denotes the on-site force (generated by a specific on-site potential).

As examined in earlier work [6], a crucial site to consider (to examine the possibility of an exact compacton solution) is the first site that is not a part of the compacton structure (and has an identically zero ordinate). One can easily verify that the existence of $u_n \equiv 0$ as a uniform steady state imposes $f(0) = 0$, and thereafter the presence of a cut-off site (after—equivalently before—which $u_n = 0, \forall n$) necessitates $g(0) = 0$. These conditions are starting points for our considerations.

Let us now follow the inverse method of [9] and seek a bright compacton solution on top of the uniform zero background, in the form

$$u_n = \sin(k(n - n_0)) \tag{2}$$

where u_0 is an integer. It is then straightforward to obtain that

$$u_{n+1} + u_{n-1} = 2\cos(k)u_n. \tag{3}$$

Hence, selecting the on-site force in the form

$$f(u_n) = -2\cos(k)u_n g(u_n) \tag{4}$$

for a generic u_n, we have satisfied identically the equations of motion (for an infinite lattice). The relevant model thus becomes

$$\ddot{u}_n = g(u_n)[u_{n+1} + u_{n-1} - 2su_n] \tag{5}$$

where g is an arbitrary nonlinear function and s can take values in the interval $[-1, 1]$ and is connected with k through $k = \arccos(s)$.

However, in this case (of an exact solution through the inverse method) there is an additional concern. Since n_0 is an integer, the solution vanishes at site n_0 (we will consider this the starting site of the compact solution to be constructed). It should also be ensured that at the other end of the compact support (say, l sites after n_0, at site $n_0 + l$), the solution also vanishes. This is imperative in order to satisfy not the equation of the site $n_0 + l$ (which is satisfied identically, if the ordinate is $u_{n_0+l} = 0$), but rather that of the site u_{n_0+l-1}. This imposes the *quantization condition*

$$kl = L\pi \tag{6}$$

where L is also an integer. This is an important condition for the existence of exact solutions which restricts the width parameter k to fractional multiples of π.

We have tested the existence of the above established solutions in numerical experiments. In all of the cases studied for bright discrete compactons, we have used the so-called Ablowitz–Ladik [10] discretization of the cubic nonlinearity, as our motivation for the choice of $g(u_n) = u_n^2$ (a choice followed for all the numerical studies of bright compactons presented herein).

For $L = 1$, we can obtain solutions with one half-period (the argument of the sine running from 0 to π). Such solutions have been identified for one to six site compactons ($l = 2$ to $l = 7$, shown in the panels of figure 1), as well as for a much larger number of sites ($l = 25$ is also shown in figure 1). In all of these cases, once the solution u_n has been verified, the numerical linear stability of the solution has been examined through the study of the eigenvalue problem

$$\lambda^2 v_n = g(u_n)[v_{n+1} + v_{n-1} - 2sv_n] \tag{7}$$

which can be restricted only to the sites $n_0 + 1, \ldots, n_0 + l - 1$; the linear stability of all other sites is a direct consequence of the algebraic structure of the evolution equations: $g(0) = g'(0) = 0$ and hence $v_n \equiv 0$ for $n \leqslant n_0$ and $n \geqslant n_0 + l$.

The surprising finding (also in comparison with earlier studies such as [6]) is that *independently of the number of sites*, the discrete compactons are found to be linearly stable.

An additional item of interest is the study of solutions which include multiple half-periods of the sine, i.e. solutions in which the sine arguments extend from 0 to 2π ($L = 2$) or from 0 to 3π ($L = 3$). These can be systematically constructed since, in general, L determines the number of half-periods covered by the argument of the sine. These solutions are even more intriguing since they can be considered as a compacton–anticompacton pair (for $L = 2$) or a 'multi-compacton' (with up-down-up-down components in analogy with solutions already examined for discrete solitons, see, e.g., [11]).

Our study indicates the interesting result that there are two categories (at least with respect to their linear stability characteristics) of solutions among the multi-compacton ones presented above. One of them consists of the 'return-to-zero' (RZ) multi-compactons in which the compacton and anti-compacton are separated by a zero site. This type of solution occurs for $L > 1$ and $l = Lr$, where r is also an integer. The other consists of the 'non-return-to-zero' (NRZ) compactons where $l = Lr + q$, where $q = 1, \ldots, L - 1$. In both cases, the expression multiplying π in the expression for k is meant to represent the relevant fraction

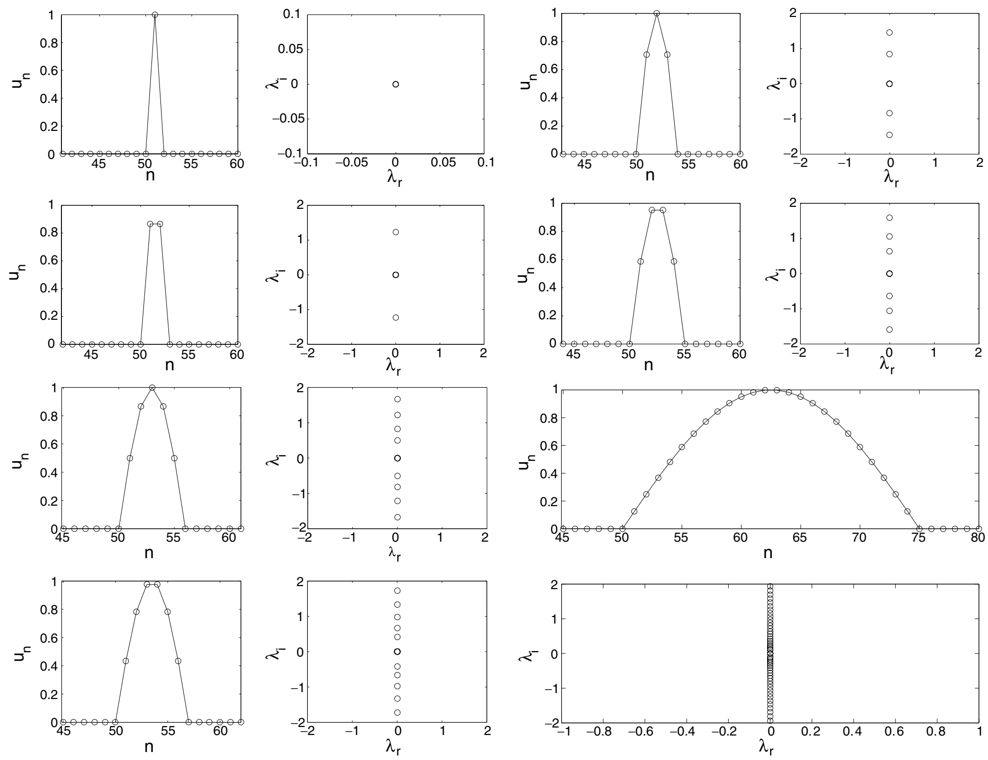

Figure 1. The top left subplot shows examples of one- and two-site compactons (top and bottom panels, respectively). The left panel shows the solution, while the right panel shows the results of the corresponding linear stability analysis. In the latter, the spectral plane (λ_r, λ_i) of the real versus imaginary part of the eigenvalue is shown. The presence of eigenvalues with nonzero real part indicates the presence of an instability. The top right subplot shows the same results for three- and four-site compactons, and the bottom left subplot for five- and six-site compactons. These cases correspond, respectively, to $l = 2, \ldots, 7$. Finally, the bottom right subplot shows the case of $l = 25$, indicating that the bright, discrete compacton is linearly stable independent of the number of sites that it comprises.

in its irreducible form. In this case the compacton and anti-compacton occur sequentially without a site of vanishing ordinate separating them. These two different classes of solutions were found to possess *different* linear stability characteristics. Figure 2 shows two examples (one RZ and one NRZ) for an up-down configuration (in the left panel) and for an up-down-up configuration (in the right panel). It is clear that the RZ multi-compactons appear to be linearly stable, whereas the NRZ have N unstable eigenmodes (where N is the number of neighbouring compacton–anticompacton pairs). The detailed study of multi-compactons (and of the relevant interaction eigenmodes between waves [11]) is an issue of importance that will be reported in a future publication[5]. We have examined here the basic features of existence and linear stability of such configurations.

[5] The multi-compacton solutions merit additional investigation not only to examine their interaction properties in analogy with what is known for discrete solitons, but also for an additional reason: our preliminary numerical investigations seem to indicate the possibility of nonlinear instabilities for such configurations *even in the case where the waves are linearly stable*. It may then be important to establish the nonlinear stability and detailed dynamical picture of the behaviour of such structures in future work.

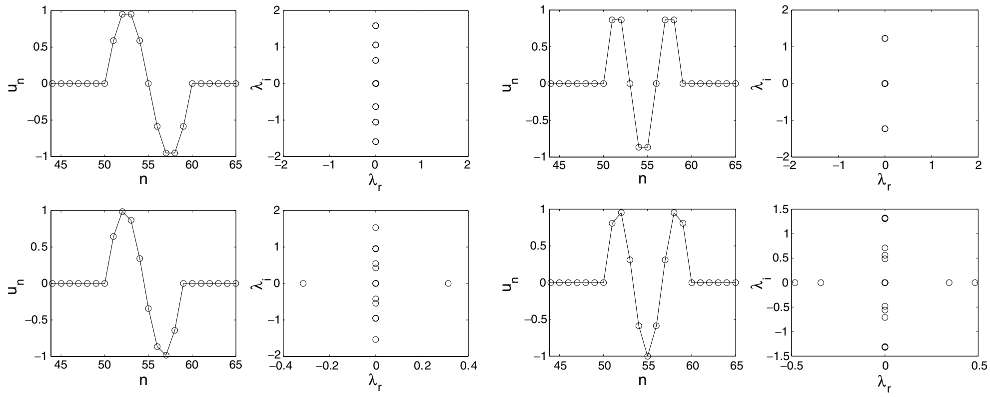

Figure 2. The left subplot shows the case of a two-compacton composite (a compacton–anti-compacton) configuration created for $L = 2$ and $k = \pi/5$ (top panel: linearly stable RZ) and $k = 2\pi/9$ (bottom panel: linearly unstable NRZ). The right subplot shows two similar results but for a three-compacton. The top RZ panel shows the case of $k = \pi/3$, while the bottom NRZ panel shows the case of $L = 3$ and $k = 3\pi/10$.

3. Trigonometric solutions: dark case

Motivated by the (trigonometric compacton) solution of equation (2), our starting point in this analysis will be equation (5). However, for dark discrete compactons to exist in this equation, asymptoting at steady states $\pm u_0$, $g(\pm u_0) = 0$ has to be satisfied. Without loss of generality, we will set $u_0 = 1$, and as the simplest example of such a nonlinear function g, we will consider

$$g(u_n) = u_n^2 - 1. \tag{8}$$

As our ansatz solution, we again select the sinusoidal one of equation (2), where n_0, if integer, is the middle site of the compacton. However, n_0 does not have to be an integer in this case (it can be, e.g., a half-integer for a wave with an even number of sites with ordinates $\neq \pm u_0$, as we will see below).

If we consider the dynamical equation (5) with the nonlinearity (8), we immediately see that the equation is identically satisfied for sites with ordinates $\pm u_0$. We then only have to consider the equation of the site just before (or just after) the steady state (the first or the last site of the compacton). This equation imposes a quantization condition

$$\sin(k(n - n_0)) = \pm u_0 \equiv \pm 1 \tag{9}$$

where for a dark compacton both the $(-)$ and the $(+)$ signs should be achieved for two different integers n_1 and n_2 such that $n_1 < n_0 < n_2$. This imposes

$$n_j = n_0 + [4m + (-1)^j]\frac{\pi}{2k} \qquad j = 1, 2 \tag{10}$$

and, hence, the quantization condition takes the form

$$k = \frac{4\pi m}{l} \qquad l = n_2 - n_1. \tag{11}$$

As mentioned above, these quantization conditions can be satisfied in more than one way (e.g., the two terms on the right-hand side of the second one of equations (11) can be both integers or both half-integers), allowing us to construct dark compactons with an arbitrary number of sites. All of these solutions were found to be linearly stable in this case also. An example

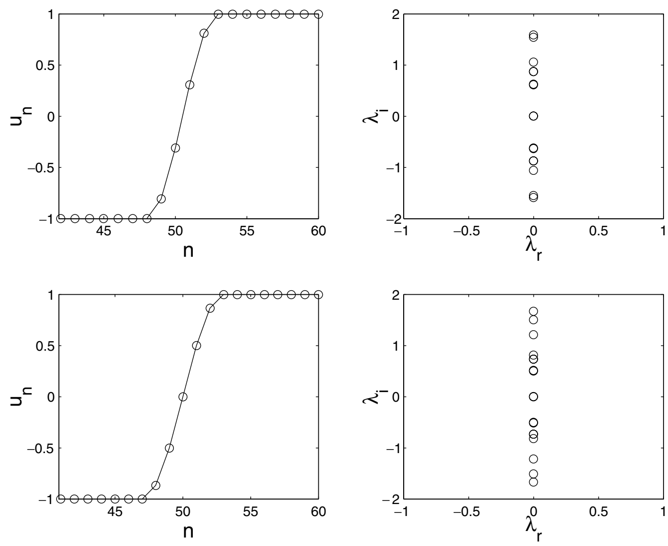

Figure 3. Dark discrete compactons: in the top panel, a four-site compacton is shown for $n_0 = 50.5$ and $k = \pi/5$; the bottom panel shows a five-site counterpart created for $n_0 = 50$ and $k = \pi/6$. As the right panels indicate, both solutions are linearly stable.

is shown in figure 3, where a compacton with four sites with $u_n \neq \pm 1$ is shown, constructed with $m = 0$, $k = \pi/5$ and $n_0 = 50.5$. In general, such 'even site' dark compactons, with $2l$ sites, can be achieved by $k = \pi/(2l + 1)$ and $n_0 = [n_0] + 1/2$ (where the brackets denote the integer part). In contrast, 'odd site' dark compactons with $2l - 1$ sites are achieved by using $k = \pi/(2l)$ and $n_0 = [n_0]$. An example for $n_0 = 50$ and $k = \pi/6$ is given in the bottom panel of figure 3.

In a way similar to what was described above for bright compactons, one can create dark multi-compacton configurations. In this case the essential difference resides in selecting n_1 according to

$$n_1 = n_0 + (4m - 3)\frac{\pi}{2k} \qquad (12)$$

and n_2 according to equation (11) for a two-compacton composite. This is shown in figure 4 for both the case with an even and the one with an odd number of sites in the two constituent waves of the multi-compacton. Similarly, selecting the initial and final values of the argument of the sine (running over many trigonometric circles), multi-compactons with three (e.g., the sine argument running from $-\pi/2$ to $5\pi/2$) or more constituent waves can be systematically achieved. It is interesting to note that in this case (of dark waves), we have found the waves to be linearly stable (see, e.g., figure 4) independent of whether the zero crossing does or does not correspond to a lattice site.

4. Can discrete compactons travel ?

We now consider the solutions presented above and examine their potential for travelling. We discuss this topic in detail for the solutions of equation (2), but the generalization of the arguments to arbitrary compactly supported envelope functions will be immediate. By 'travelling wave' we mean a solution depending on $kn - X(t)$, where $X(t)$ is a continuously differentiable function of time t.

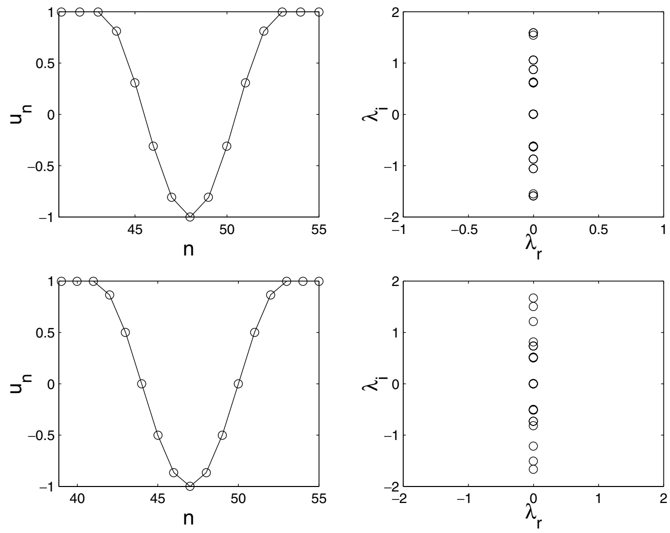

Figure 4. Composite of two dark compactons (of the two varieties presented in the previous picture: even-site compactons with no site at 0 and odd-site compactons with the middle site at 0) to create a dark multi-compacton. The resulting configuration is, in both cases, linearly stable.

For the exact travelling generalization of equation (2), the solution (now time-dependent) profile will be

$$u_n = \sin(k(n - n_0) - X(t)). \tag{13}$$

The key insight for the existence of genuine travelling solutons stems, we believe, from the quantization condition which should be satisfied $\forall t \in R$. This imposes the condition

$$k(n - n_0) - X(t) = L\pi. \tag{14}$$

Hence $n = n_0 + L\pi/k + X(t)/k$ must be integer for every moment in time. Suppose that we consider a time t_0 for which $n_0 + L\pi/k + X(t_0)/k = N_0 \in Z$. Then if we look at time $t_0 + \epsilon$, where ϵ is chosen such that it is small enough, i.e. $X(t) = X(t_0) + \epsilon X'(t_0) + O(\epsilon^2)$, and $\epsilon X'(t_0)/k$ is not an integer (which is always possible since ϵ is arbitrary), then the quantization condition is violated.

In view of the above argument, the restriction that the quantization condition imposes on the solution *cannot be satisfied at every moment in time*. As a result, we conclude that genuinely travelling discrete compactons cannot exist. Similar considerations can immediately be applied in the case of dark discrete compactons. More generally, such arguments can be applied to any function of the form $u_n = H(k(n-n_0) \pm \Omega t)$. A similar quantization condition will be applied equating the argument of H with s such that $H(s) = 0$. Then a similar argument to the one above (with $L\pi$ replaced by s) will be applied, disallowing the possibility of discrete travelling compactons (at least for sufficiently smooth envelope functions with a discrete set of zero crossings).

5. Generalizations

Let us now generalize the approach developed in the previous sections, considering the bright case as an example. Equation (5) can be viewed as an evolution system with nonlinear dispersion (a necessary ingredient for the existence of compactons). The lattice with a generalized nonlinear dispersion

$$\ddot{u}_n = g(u_n)P_M(u_n) \qquad P_M(u_n) \equiv \frac{1}{2}\sum_{j=0}^{M} c_j[u_{n-j} + u_{n+j}] \tag{15}$$

(where $g(u_n)$ satisfies the same conditions as in section 2 (in particular, $g(0) = 0$) and c_j are real coefficients) admits static compacton solutions for definite relations among $M + 1$ coefficients c_j $(j = 0, 1, \ldots, M)$, which are to be specified in what follows. To this end we consider the system

$$\begin{cases} P_M(u_n) = 0 & \text{if} \quad \tilde{n} \leqslant n \leqslant \tilde{n} + l \\ u_n = 0 & \text{if} \quad n < \tilde{n} \quad \text{and} \quad n > \tilde{n} + l \end{cases} \tag{16}$$

with the matching conditions $u_{\tilde{n}} = u_{\tilde{n}+l} = 0$. A solution of (16) will, clearly, be at the same time a stationary solution of (15). The latter can be represented in the form

$$u_n = \chi_l(n - \tilde{n})v_{n-\tilde{n}} \qquad \chi_l(n) = \begin{cases} 1 & \text{if} \quad 0 \leqslant n \leqslant l \\ 0 & \text{if} \quad n < 0 \quad \text{and} \quad n > l. \end{cases} \tag{17}$$

Then (16) is reduced to the system of $l - 1$ linear algebraic equations

$$\sum_{j=0}^{M_1} c_j v_{n+j} + \sum_{j=0}^{M_2} c_j v_{n-j} = 0 \tag{18}$$

where $n = 1, \ldots, l - 1, M_1 = \min\{M, l - n\}$; $M_2 = \min\{M, n\}$ and v_n admits the representation

$$v_n = \sum_{k=1}^{l-1} b_k \sin\left(\frac{\pi}{l}kn\right) \tag{19}$$

where b_k are constants. System (18) relates the two sets of parameters: $c = \{c_0, c_1, \ldots c_M\}$ and $v = \{b_1, \ldots b_{l-1}\}$. It can be viewed either as a homogeneous algebraic system with respect to b (when $l \geqslant M + 2$), allowing one to find $b(c)$, or as a homogeneous algebraic system with respect to c (when $l \leqslant M + 2$) which allows one to find $c(b)$. In the first case, $b(c)$ corresponds to the direct problem, in which the solution is obtained for a given type of equation, while in the second case, $c(b)$ yields the reconstruction of the potential starting from the given solution.

An interesting and natural generalization of the inverse problem for trigonometric function solutions is provided by the choice of elliptic functions as the original ansatz for the compactly supported wave. This generalization is naturally motivated by the fact that using the value of the elliptic modulus as an interpolation parameter, the elliptic functions interpolate between the case of trigonometric localization over the support (for vanishing modulus) and that of exponential localization (for unit modulus). It is therefore of interest to observe how this natural parameter affects the stability and dynamics of the discrete compactons. We will restrict the consideration to the case $M = 2$.

We now use as our starting point ansatz the form

$$u_n = \mathrm{sn}(k(n - n_0)|m). \tag{20}$$

Here sn is the Jacobi elliptic function which becomes identical to sine for $m = 0$ (m is the modulus of the elliptic function, assuming values between 0 and 1). Using this ansatz, we obtain that

$$u_{n+1} + u_{n-1} = 2\mathrm{cn}(k)\,\mathrm{dn}(k)\frac{u_n}{1 - m\mathrm{sn}^2(k)u_n^2}. \tag{21}$$

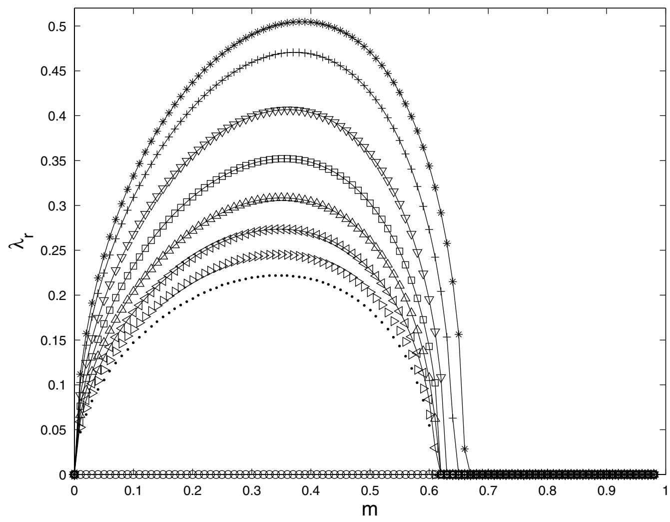

Figure 5. For the case of the elliptic function generalization of the model and for compactons with different numbers of sites, the trajectory of the eigenvalue with the maximal real part is shown: the real part of the relevant eigenvalue (instability growth rate) as a function of the elliptic modulus m. The circles show the (always stable) one-site compacton. The stars show the two-site compacton, the plus symbols the three-site case, the down triangles the four-site one, the squares correspond to five-site compacton, the up triangles to six-site compacton, the left triangles to seven-site compacton and the right ones to eight-site compacton, while the dots correspond to nine-site compacton. We can see that in all cases (apart from the one-site one) the compacton is unstable for small m but becomes stable for large values of $m > m_c$. The growth rate of the relevant instability is larger for smaller compacton widths.

Hence, in this case the analogue of equation (5) will be the dynamical evolution equation

$$\ddot{u}_n = g(u_n)\left[u_{n+1} + u_{n-1} - 2\mathrm{cn}(k)\,\mathrm{dn}(k)\frac{u_n}{1 - m\,\mathrm{sn}^2(k)u_n^2}\right]. \tag{22}$$

In this case, the quantization condition will become

$$n = n_0 + L\frac{2K(m)}{k} \equiv n_0 + l \tag{23}$$

where $K(m)$ is the complete elliptic integral of the first kind. Once again, depending on the values of l and L, one can create single compactons with different numbers of sites as well as multi-compactons. Such solutions occur for choices of k exactly analogous to the ones made in section 2, but with π substituted by $2K(m)$.

A natural question concerns the stability of such compact structures as a function of m. We know that all of them, except for NRZ multi-compactons, are (linearly) stable for $m = 0$. Here, numerical investigation produces an interesting result. It turns out that apart from single-site compactons (which are generically stable), linear stability of multi-site configurations strongly depends on the value of the modulus and on the number of sites participating in the wave (if $L\frac{2K(m)}{k} = l \in Z$, we say, by convention, that $l - 1$ sites participate in the compacton). Figure 5 then shows that, even though multi-site compactons are marginally stable for $m = 0$, they become *immediately* unstable for $m > 0$ and remain unstable up to a critical m_c which is (weakly) dependent (typically $m_c \in (0.6, 0.7)$) on the number of sites of the compacton. Furthermore, the growth rate of the instability (when the wave is unstable) depends significantly

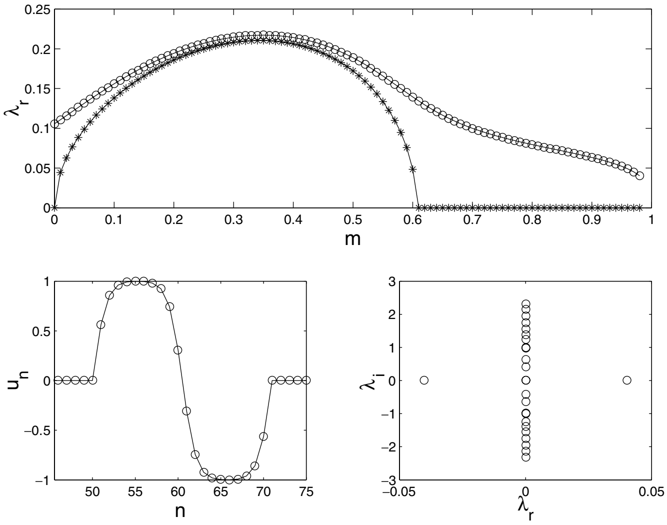

Figure 6. The figure examines the possibility of the elliptic modulus playing a stabilizing role for an unstable (for $m = 0$) solution. For a two-compacton solution with $l = 21$, the real part of the two most unstable eigenvalues is shown as a function of the elliptic function modulus m. We see that initially (for small m), the solution becomes even more unstable than for $m = 0$ (in agreement with the findings of figure 5); however, for large m (as $m \to 1$) the instability rate decreases and eventually becomes less than the corresponding rate at $m = 0$. The bottom panel shows the solution and its linear stability analysis for $m = 0.99$.

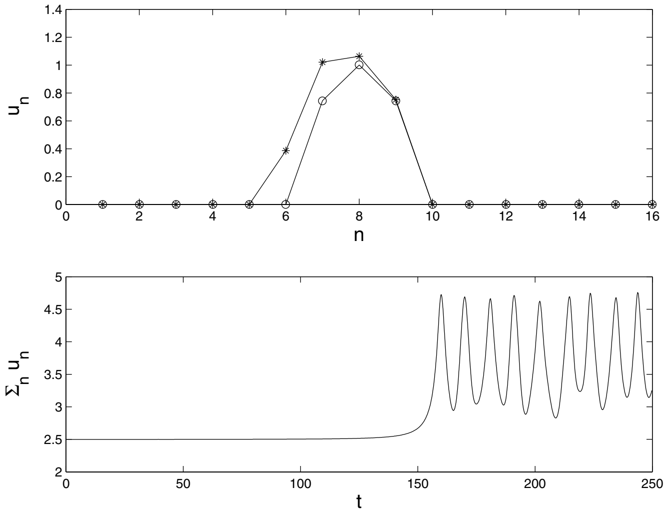

Figure 7. The dynamical manifestation of the instability of an initial configuration of a three-site unstable compacton for $m = 0.35$ (the corresponding profile is denoted by the circles). This profile is reshaped into an oscillatory one (a compactly supported breathing profile) as time evolves. The final configuration at $t = 250$ is shown by stars in the top panel, while a diagnostic (the sum of all the ordinates $\sum_n u_n$) indicating the oscillatory nature of the resulting configuration as a function of time is shown in the bottom panel.

on the value of the modulus and on the number of sites in the compacton. The structure is most unstable for compactons with fewer sites (systematically a larger instability growth rate for two-site compactons than for three-site ones, and so on, but with the above-mentioned exception of the always stable one-site compactons). For $m > m_c$, the structure is always linearly stable.

We also examined whether linearly unstable structures such as NRZ multi-compactons (for $m = 0$) can be stabilized through the increase of m. A typical example of such investigations is shown in figure 6 for $l = 21$. We observe that even though the growth rate increases for small m and subsequently decreases for larger values of the modulus, the structure is not (linearly) restabilized. The growth rate of the instability does become, however, lower than the one for the $m = 0$ case, as $m \to 1$.

Finally, we examined the manifestation of the instability of a discrete elliptic function compacton, for $m > 0$. In particular, figure 7 shows the dynamical evolution of the instability of a three-site compacton for $m = 0.35$. The top panel shows the initial and final configurations (after time $t = 250$), while the bottom panel shows a diagnostic of the solution indicating that the profile reshapes itself through the instability into an oscillatory ('breathing' in time) configuration. The initial configuration of the three-site compacton was perturbed by a random (uniformly distributed) noise of amplitude 10^{-3} to observe this time evolution.

6. Conclusions

In this letter, we have examined the possibility of bright, as well as dark, discrete compactons existing in a class of lattice models, as *exact solutions* of the dynamical equations of motion. To analyse this possibility, we used an inverse method (i.e. postulate the ansatz solution, find the model) motivated by the works of [8, 9], for the appropriate trigonometric and elliptic function ansätze. However, for compactly supported solutions we argued that the extra aspect of a quantization condition is necessary to ensure the existence of such an exact compacton solution. Different quantization conditions are, of course, imposed to obtain bright and dark solutions. Moreover, this methodology can be systematically generalized to obtain multi-compacton solutions (of alternating parity), upon appropriate selection of the interval over which the argument of the envelope extends. We also examined the linear stability of such multi-compacton configurations. The quantization condition for the envelope function was used to argue against the possibility for *exact* discrete travelling waves with compact support. The generalization of the trigonometric case to an elliptic function ansatz presented us with the possibility of a one-parameter variation to examine the stability and dynamics of the solutions as a function of the modulus of the elliptic function m. We found in this way that discrete compactons are marginally stable in the trigonometric case ($m = 0$) and become immediately unstable as m becomes positive, but that the instability is saturated for sufficiently large m ($\approx$0.6–0.7).

There are still a number of outstanding questions regarding these discrete compactly supported waves. It would be interesting to know whether these structures encounter a form of Peierls–Nabarro barrier in their motion through the lattice, and then to quantify this barrier, which we may expect to be different from the case of regular intrinsic localized modes. Another issue of interest would be to examine in more detail the multi-compacton solutions and their dynamics in comparison with their regular discrete soliton counterparts.

Finally, in all the cases that we are currently aware of, discrete compactons exist in models with nearest neighbour interactions. It would be interesting to examine in more detail whether models with longer range interactions can admit solutions of this type. An obvious issue in that direction is the existence of conditions such as the quantization condition explored herein.

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